23. *Infinite potential well with a barrier*

Consider a quantum particle in one-dimensional space in the presence of the potential

\[ U(x) = \begin{cases} \infty, & |x| > a \\ U_1 \delta(x), & |x| \leq a \end{cases} \]  

with \( a, U_1 > 0 \).

(a) Sketch \( U(x) \) as a function of \( x \) in a diagram.

(b) Make an appropriate ansatz for the eigenfunctions and formulate conditions to determine the coefficients.

(c) Derive equations for the energy eigenvalues \( \varepsilon \) of even and odd eigenfunctions and describe the graphical solution of the resulting transcendental equations.

(d) Determine the energy eigenvalues and sketch the eigenfunctions in the limiting cases \( U_1 \rightarrow 0 \) and \( U_1 \rightarrow \infty \). Discuss the possibility of degeneracy.

24. *Numerical solution of the Schrödinger equation*

In order to numerically solve the one-dimensional Schrödinger equation \( \phi''(x) = (U(x) - \varepsilon)\phi(x) \) one chooses a step width \( \Delta x > 0 \) and then introduces \( \phi_n := \phi(x_n), x_n := n\Delta x, n \in \mathbb{Z} \).

(a) Show that

\[ \phi''(x_n) = \frac{\phi_{n+1} - 2\phi_n + \phi_{n-1}}{(\Delta x)^2} + \mathcal{O}((\Delta x)^2) \]  

and express the Schrödinger equation in the form

\[ \phi_{n+1} = f(\phi_n, \phi_{n-1}) + (\Delta x)^2 g(\phi_n, x_n, \varepsilon) + \mathcal{O}((\Delta x)^4). \]  

(b) By implementing Eq. (3) in a computer program, determine approximately the eigenvalues \( \varepsilon \) of the bound states for the potential \( U(x) = -\frac{6}{\cosh^2 x} \), which was discussed in problem 17 (sheet no. 7).

(Hints: Neglect in Eq. (3) the terms \( \mathcal{O}((\Delta x)^4) \). Make use of the symmetry of the wave functions as well as of the boundary conditions \( \phi(x \rightarrow \pm \infty) \rightarrow 0 \).)

(c) Plot the numerically determined approximate eigenfunctions \( \phi_n \) and the exact eigenfunctions \( \phi(x_n) \) obtained in problem 17 in one diagram and compare them.
25. **Scattering in a one-dimensional potential**

Consider a quantum particle with energy $\varepsilon > U_0$ in one-dimensional space which is scattered at the potential

$$U(x) = \frac{U_0}{1+e^{-x}}, \quad U_0 > 0.$$  \hfill (4)

(a) Sketch the potential $U(x)$.

(b) Determine the form of the wave function $\psi(x)$ in the limits $x \to -\infty$ and $x \to +\infty$. Express the transmission and the reflection coefficient $T$ and $R$ in terms of the parameters appearing in your ansatz for $\psi(x)$.

(Note: The information about the shape of $U(x)$ is encoded in the yet unknown coefficients. These must be determined from the actual solution of the Schrödinger equation, which is done in the following steps.)

(c) Apply the substitution $z = -e^{-x}$ in order to transform the Schrödinger equation for $\psi(x)$ into a differential equation for $\psi(z)$. Determine the singularities of this differential equation, i.e., the points $z$ near which the solution $\psi(z)$ asymptotically behaves as $(z - \zeta)^\lambda$ with $\lambda \in \mathbb{C}$, $\zeta \in \mathbb{R}$. Make use of your results in part (b) in order to choose among the possible solutions for $\lambda$ the appropriate ones.

(d) Based on your considerations in part (c), choose a reasonable ansatz for the form of $\psi(z)$ in order to bring the transformed Schrödinger equation into the form of the hypergeometric differential equation

$$z(z-1)\phi''(z) + [(a + b + 1)z - c]\phi'(z) + ab\phi(z) = 0$$ \hfill (5)

for a new function $\phi(z)$. Determine the parameters $a$, $b$, and $c$ in terms of $U_0$ and the energy eigenvalue $\varepsilon$ appearing in the original Schrödinger equation.

(e) **[Bonus problem]** Equation (5) is solved by the hypergeometric function $F(a,b,c|z) \equiv {}_2F_1(a,b,c|z)$ (see, e.g., Abramowitz & Stegun, “Handbook of Mathematical Functions”, ch. 15). By making use of the results in part (b), identify the reflection and the transmission coefficient from the asymptotic behavior of $\psi(z(x))$. 
