29. Ehrenfest’s theorem in the Heisenberg picture

Consider a quantum particle of mass \( m \) in one-dimensional space governed by the (time-independent) Hamilton operator \( \hat{H} = \frac{1}{2m} \hat{p}^2 + V(\hat{x}) \), where \( \hat{x} \) and \( \hat{p} \) are the position and the momentum operator in the Schrödinger picture and the external potential \( V \) is an analytic function.

(a) Let \( \hat{A} \) and \( \hat{B} \) be two operators fulfilling the commutator relation \([\hat{A}, \hat{B}] = 1\) and let \( f : \mathbb{C} \to \mathbb{C} \) be an analytic function. Show that \([f(\hat{A}), \hat{B}] = f'(\hat{A})\), where \( f' \) is the derivative of \( f \).

(Hint: Use ex. 27(a) of homework no. 11.)

(b) Calculate the commutators \([\hat{H}, \hat{x}(\hat{H})(t)]\) and \([\hat{H}, \hat{p}(\hat{H})(t)]\), where an operator \( \hat{A}(\hat{H})(t) := \hat{U}(t)\dagger \hat{A} \hat{U}(t) \) in the Heisenberg picture is related to the corresponding operator \( \hat{A} \) in the Schrödinger picture via the time evolution operator \( \hat{U}(t) \) (see lecture).

(c) Evaluate Heisenberg’s equations of motion \( \frac{d}{dt} \hat{A}(\hat{H})(t) = \frac{i}{\hbar} [\hat{H}, \hat{A}(\hat{H})(t)] \) for the position and momentum operator and, based on this, derive Ehrenfest’s theorem (see ex. 11 of homework no. 4).

30. Expectation values in the Heisenberg picture

Using the results of exercise 29, solve Heisenberg’s equations of motion for the position operator \( \hat{x}(\hat{H})(t) \) and the momentum operator \( \hat{p}(\hat{H})(t) \) for

(a) a free particle \( (V(\hat{x}) = 0) \), and

(b) a one-dimensional harmonic oscillator \( (V(\hat{x}) = \frac{1}{2}m\omega^2 \hat{x}^2) \).

Calculate and discuss the expectation values \( \langle \hat{x} \rangle_t \) and \( \langle (\hat{x} - \langle \hat{x} \rangle)_t \rangle^2 \) at time \( t \geq 0 \) for both cases (a) and (b), provided the quantum state at time \( t = 0 \) is given in position space by the wave function

\[
\psi(x, t = 0) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{4\sigma^2} + ikx\right),
\]

with \( \sigma > 0 \) and \( k \in \mathbb{R} \).
31. **Minimal wave packet**

For two Hermitian operators $\hat{A}$ and $\hat{B}$ the Heisenberg uncertainty relation

$$\Delta \hat{A} \Delta \hat{B} \geq \frac{1}{2} |\langle [\hat{A}, \hat{B}] \rangle|,$$

(2)

with $\Delta \hat{C} := \sqrt{\langle (\hat{C} - \langle \hat{C} \rangle)^2 \rangle}$, has been proven in the lecture. A state $|\psi\rangle$ is called a **minimal wave packet** if in Eq. (2) equality holds.

(a) Show that a state $|\psi\rangle$ is a minimal wave packet if

$$(\hat{A} - \langle \hat{A} \rangle)|\psi\rangle = i r (\hat{B} - \langle \hat{B} \rangle)|\psi\rangle \quad \text{or} \quad (\hat{B} - \langle \hat{B} \rangle)|\psi\rangle = i r (\hat{A} - \langle \hat{A} \rangle)|\psi\rangle$$

(3)

with a real number $r \in \mathbb{R}$ applies.

(b) Consider spatial dimension $d = 1$. Formulate Eq. (3) for $\hat{A} := \hat{x}$ and $\hat{B} := \hat{p}$ in position space representation and solve the resulting differential equation for the wave function $\psi(x, t)$ with the ansatz $\psi(x, t) =: \exp(\phi(x, t))$.

(c) Using the results of part (b), check whether the Gaussian wave packet of a free particle (see lecture)

$$\psi(x, t) = \frac{1}{(2\pi \Lambda(t))^{1/4}} \exp \left( - \frac{(x - \frac{hq}{m} t)^2}{4\Lambda(t)} + i \left( qx - \frac{hq^2}{2m} t \right) \right), \quad \Lambda(t) = \sigma^2 + \frac{\hbar}{2m} t$$

(4)

with $\sigma > 0$ and $q \in \mathbb{R}$ is a minimal wave packet.

(d) Consider a one-dimensional harmonic oscillator of frequency $\omega > 0$ in the coherent state with parameter $\gamma \in \mathbb{C}$ (see ex. 21 of homework no. 9). The corresponding wave function $\psi_\gamma(x, t)$ fulfills $\hat{b} \psi_\gamma(x, t) = \gamma \exp(-i \omega t) \psi_\gamma(x, t)$ with the annihilation operator

$$\hat{b} = \sqrt{\frac{m}{2\hbar}} \frac{d}{dx} + \sqrt{\frac{\hbar}{2m\omega}} x.$$ 

Check, with the help of the results of part (b), whether the coherent state with parameter $\gamma \in \mathbb{C}$ is a minimal wave packet.