Consider a point \( \mathbf{r} \) in three-dimensional space \( \mathbb{R}^3 \) described by spherical coordinates \((r, \vartheta, \varphi)\) and define the vectors

\[
\mathbf{e}_r := \frac{\partial}{\partial r} / \left| \frac{\partial}{\partial r} \right|, \quad \mathbf{e}_\vartheta := \frac{\partial}{\partial \vartheta} / \left| \frac{\partial}{\partial \vartheta} \right|, \quad \mathbf{e}_\varphi := \frac{\partial}{\partial \varphi} / \left| \frac{\partial}{\partial \varphi} \right|.
\]

(a) Show that, with respect to the Cartesian standard basis \((\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)\), one has:

\[
\begin{align*}
\mathbf{e}_r &= \sin(\vartheta) \cos(\varphi) \mathbf{e}_x + \sin(\vartheta) \sin(\varphi) \mathbf{e}_y + \cos(\vartheta) \mathbf{e}_z, \\
\mathbf{e}_\vartheta &= \cos(\vartheta) \cos(\varphi) \mathbf{e}_x + \cos(\vartheta) \sin(\varphi) \mathbf{e}_y - \sin(\vartheta) \mathbf{e}_z, \\
\mathbf{e}_\varphi &= -\sin(\varphi) \mathbf{e}_x + \cos(\varphi) \mathbf{e}_y.
\end{align*}
\]

(b) Show that \(\mathbf{e}_r, \mathbf{e}_\vartheta,\) and \(\mathbf{e}_\varphi\) form an orthonormal right-handed basis, i.e., \(\mathbf{e}_\alpha \cdot \mathbf{e}_\beta = \delta_{\alpha\beta}\) with \(\alpha, \beta \in \{r, \vartheta, \varphi\}\), as well as \(\mathbf{e}_r \times \mathbf{e}_\vartheta = \mathbf{e}_\varphi\).

(c) Expand all partial derivatives \(\frac{\partial \mathbf{e}_\alpha}{\partial \beta}\) with \(\alpha, \beta \in \{r, \vartheta, \varphi\}\) in the basis \((\mathbf{e}_r, \mathbf{e}_\vartheta, \mathbf{e}_\varphi)\).

(d) The differential \(d\psi(\mathbf{r})\) of a scalar field \(\psi : \mathbb{R}^3 \to \mathbb{C}\) at point \(\mathbf{r} = r \mathbf{e}_r \in \mathbb{R}^3\) can be expressed in terms of the differential \(dr\) and the gradient \(\nabla \psi(\mathbf{r})\) as \(d\psi(\mathbf{r}) = d\mathbf{r} \cdot \nabla \psi(\mathbf{r})\). Since \(\nabla \psi(\mathbf{r})\) is a vector, it can be expanded in the basis \((\mathbf{e}_r, \mathbf{e}_\vartheta, \mathbf{e}_\varphi)\): \(\nabla \psi(\mathbf{r}) = a_r \mathbf{e}_r + a_\vartheta \mathbf{e}_\vartheta + a_\varphi \mathbf{e}_\varphi\), where the coefficients \(a_r, a_\vartheta, a_\varphi\) may depend on the scalar field \(\psi\) as well as on the point \(\mathbf{r}\). Determine the coefficients \(a_r, a_\vartheta, a_\varphi\) by expressing \(d\psi(\mathbf{r})\) in spherical coordinates and expanding \(dr\) in the basis \((\mathbf{e}_r, \mathbf{e}_\vartheta, \mathbf{e}_\varphi)\).

In this way, show that the nabla operator \(\nabla\) in spherical coordinates is given by

\[
\nabla = \mathbf{e}_r \frac{\partial}{\partial r} + \frac{1}{r} \mathbf{e}_\vartheta \frac{\partial}{\partial \vartheta} + \frac{1}{r \sin(\vartheta)} \mathbf{e}_\varphi \frac{\partial}{\partial \varphi}
\]

and determine the Laplace operator \(\Delta := \nabla \cdot \nabla\) in spherical coordinates.

(e) Determine the representations of the angular momentum operator \(\hat{\mathbf{L}} = \mathbf{r} \times (-i\hbar \nabla)\), its Cartesian components \(\hat{L}_\alpha = \mathbf{e}_\alpha \cdot \hat{\mathbf{L}}\) with \(\alpha \in \{x, y, z\}\), as well as the operator \(\hat{\mathbf{L}}^2\) in spherical coordinates.

(f) Show the relation \(\Delta = \frac{\partial}{\partial r^2} + \frac{2\partial}{r \partial r} - \frac{1}{\hbar^2 r^2} \hat{\mathbf{L}}^2\).

please turn over
Consider a quantum system whose states are elements of a two-dimensional Hilbert space \( \mathfrak{u} \). The measurement quantities \( \mathfrak{A}, \mathfrak{B}, \) and \( \mathfrak{C} \) are described by linearly independent (Hermitian) operators \( \hat{A}, \hat{B}, \) and \( \hat{C} \), which act on \( \mathfrak{u} \) and have the orthonormal eigenstates \( |a_j\rangle, |b_j\rangle, \) and \( |c_j\rangle \) \((j \in \{1, 2\})\). Die corresponding eigenvalues \( a_j, b_j, \) and \( c_j \) take on the values \( a_1 = b_1 = c_1 = 1, \) and \( a_2 = b_2 = c_2 = -1. \)

(a) Assume that the following is known: if the system is in a state \( |\xi\rangle \in \{|a_1\rangle, |a_2\rangle\} \), it transitions after a measurement of \( \hat{B} \) (i.e., of the associated measurement quantity \( \mathfrak{B} \)) with equal probability into one of the eigenstates of \( \hat{B} \).
Determine the representation of the normalized eigenstates \( |b_j\rangle \) as well as the matrix representation of the operator \( \hat{B} \) in the basis \( |a_j\rangle \).

(Result: \( |b_j\rangle = \frac{1}{\sqrt{2}} (|a_1\rangle \pm |a_2\rangle) \), \( \hat{B}_{jk} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_{jk} \).

(b) [Bonus problem] Assume furthermore that the following is known: if the system is in a state \( |\xi\rangle \in \{|a_1\rangle, |a_2\rangle, |b_1\rangle, |b_2\rangle\} \), it transitions after a measurement of \( \hat{C} \) with equal probability into one of the eigenstates of \( \hat{C} \).
Determine the representation of the normalized eigenstates \( |c_j\rangle \) as well as the matrix representation of the operator \( \hat{C} \) in the basis \( |a_j\rangle \).

(Result: \( |c_j\rangle = \frac{1}{\sqrt{2}} (|a_1\rangle \pm i|a_2\rangle) \), \( \hat{C}_{jk} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}_{jk} \).

(c) Consider now a combined system, the states of which are elements in the product space \( \mathfrak{u}^{(12)} = \mathfrak{u}^{(1)} \otimes \mathfrak{u}^{(2)} \). Using the basis \( |a_ja_k\rangle := |a_j^{(1)}\rangle \otimes |a_k^{(2)}\rangle, j, k \in \{1, 2\} \), a generic state \( |\Psi\rangle \in \mathfrak{u}^{(12)} \) can be represented as \( |\Psi\rangle = \alpha |a_1a_1\rangle + \beta |a_1a_2\rangle + \gamma |a_2a_1\rangle + \delta |a_2a_2\rangle \) with suitable \( \alpha, \beta, \gamma, \delta \in \mathbb{C} \).

i. Determine the probability that a simultaneous measurement of \( \hat{B}^{(1)} \equiv \hat{B}^{(1)} \otimes 1^{(2)} \) and \( \hat{A}^{(2)} \equiv 1^{(1)} \otimes \hat{A}^{(2)} \) of the state \( |\Psi\rangle \) yields the values \( b_1 = b_1 \) and \( a_2^{(2)} = a_2^{(2)} \).

ii. [Bonus problem] Determine the probability that the sole measurement of \( \hat{A}^{(2)} \) of the state \( |\Psi\rangle \) yields the value \( a_2^{(2)} = a_2^{(2)} \).

(d) Consider the state \( |\psi\rangle = \frac{1}{\sqrt{2}} (|a_1a_2\rangle - |a_2a_1\rangle) \in \mathfrak{u}^{(12)}. \)

i. Show that \( |\psi\rangle \) is not a product state, i.e., it is not possible to express it as \( |\psi\rangle = |\phi^{(1)}\rangle \otimes |\phi^{(2)}\rangle \) in terms of two states \( |\phi^{(j)}\rangle \in \mathfrak{u}^{(j)} \). States \( |\psi\rangle \) of this kind are called entangled.

ii. Show that \( \langle\psi|\hat{O}^{(1)}\hat{O}^{(2)}|\psi\rangle = -1 \) holds for all \( \hat{O} \in \{\hat{A}, \hat{B}, \hat{C}\}, \) where \( \hat{O}^{(1)}\hat{O}^{(2)} \equiv (\hat{O}^{(1)} \otimes 1^{(2)})(1^{(1)} \otimes \hat{O}^{(2)}). \) Interpret this result. How do product states differ in this respect?