34. Commutators
Calculate the following commutators in position space representation:

(a) $[\hat{L}_\alpha, r_\beta]$
(b) $[\hat{L}_\alpha, V(|r|)]$
(c) $[\hat{L}_\alpha, \hat{p}_\beta]$
(d) $[\hat{L}_\alpha, \hat{p}^2]$
(e) $[\hat{L}_\alpha, \hat{L}_\beta]$

with $\alpha, \beta \in \{x, y, z\}$ and an arbitrary differentiable function $V : [0, \infty) \rightarrow \mathbb{C}$.

35. Expectation values
Consider a quantum mechanical angular momentum $\hat{L} = \hbar \hat{J}$.

(a) Given the angular momentum is in the eigenstate $|j, m\rangle$ of the operators $\hat{J}^2$ and $\hat{J}_z$, calculate the expectation values $\langle \hat{J}_x \rangle$, $\langle \hat{J}_y \rangle$, $\langle \hat{J}_y \rangle$, $\langle \hat{J}_z \rangle$ and $\langle \hat{J}_z^2 \rangle$.

(b) For $j = 1$, express the eigenstates of the operator $\hat{J}_x$ in the standard basis of eigenstates $|j, m\rangle$ of the operators $\hat{J}^2$ and $\hat{J}_z$. 

please turn over
36. **Orbitals**

The first spherical harmonics \( Y_{l,m} : [0, \pi] \times [0, 2\pi] \to \mathbb{C} \) for \( l \in \mathbb{N}, m \in \mathbb{Z}, |m| \leq l \) are given by

\[
Y_{0,0}(\theta, \phi) = \frac{1}{\sqrt{4\pi}}, \quad Y_{1,0}(\theta, \phi) = \sqrt{\frac{3}{4\pi}} \cos(\theta), \quad Y_{1,\pm1}(\theta, \phi) = \pm \sqrt{\frac{3}{8\pi}} \sin(\theta) \exp(\pm i\phi).
\]

The following linear combinations of spherical harmonics lead to real-valued functions:

\[
s := Y_{0,0}, \quad p_x := \frac{-1}{\sqrt{2}}(Y_{1,1} - Y_{1,-1}), \quad p_y := \frac{-1}{\sqrt{2}}(Y_{1,1} + Y_{1,-1}), \quad p_z := Y_{1,0}.
\]

They are called \( s \)-, \( p_x \)-, \( p_y \)-, and \( p_z \)-orbitals, respectively.

(a) Express the \( s \)- and the three \( p \)-orbitals in terms of \( \theta \) and \( \phi \).

(b) Sketch the absolute values of the \( s \)- and the three \( p \)-orbitals in three-dimensional polar diagrams.

37. **Hydrogen atom**

The wave function of an electron in a non-relativistic hydrogen atom is of the form

\[
\psi_{n,l,m}(r, \theta, \phi) = f_{n,l}(r)Y_{l,m}(\theta, \phi),
\]

where \( n, l, m \in \mathbb{Z}, n \geq 1, 0 \leq l < n, |m| \leq l \) (see lecture), with the following lowest radial parts:

\[
f_{1,0}(r) = 2a_0^{-3/2} \exp(-r/a_0), \quad f_{2,0}(r) = (2a_0)^{-3/2} \left(2 - \frac{r}{a_0}\right) \exp(-r/(2a_0)), \quad f_{2,1}(r) = 24^{-1/2}a_0^{-3/2} \frac{r}{a_0} \exp(-r/(2a_0)).
\]

Calculate and sketch in a diagram the radial probability density \( P_{n,l,m}(r) \) to find an electron in state \((n, l, m)\) at a distance \( r \geq 0 \) from the nucleus for \( n \leq 2 \). Keep in mind that \( P_{n,l,m}(r) \) is independent of the angles \( \theta \) and \( \phi \)!